Munich Center for Machine Learning (MCML)
TUM School of Computation, Information and Technology
Technical University of Munich

Identifying Total Causal Effects in Linear Models under Partial Homoscedasticity

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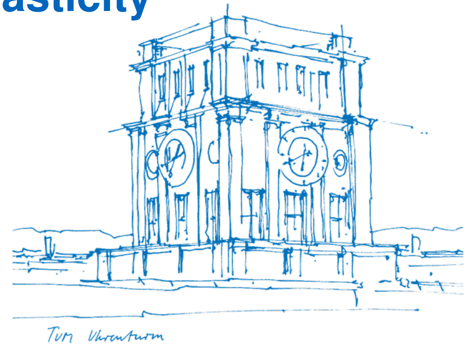

Starting point

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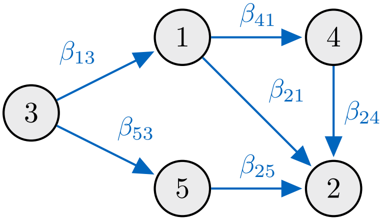
Research question: What is the total causal effect of X_i on X_j? Confidence?

- Only observational data available.
- Underlying causal structure is unknown.

Starting point
Underlying Linear SCM with Gaussian errors

Example:

$$
\begin{aligned}
X_1 &= \beta_{13} X_3 + \varepsilon_1 \\
X_2 &= \beta_{21} X_1 + \beta_{24} X_4 + \beta_{25} X_5 + \varepsilon_2 \\
X_3 &= \varepsilon_3 \\
X_4 &= \beta_{41} X_1 + \varepsilon_4 \\
X_5 &= \beta_{25} X_3 + \varepsilon_5
\end{aligned}
$$



where $\varepsilon_j \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_j^2)$

Starting point

Underlying Linear SCM with Gaussian errors

Example: Causal effect $\mathcal{C}(1 \to 2) := \frac{\mathrm{d}}{\mathrm{d}x_1}\mathbb{E}[X_2 | \,\mathrm{do}(X_1 = x_1)] = \varphi(G, \Sigma)$.

$$
\begin{aligned}
X_1 &= \beta_{13} X_3 + \varepsilon_1 \\
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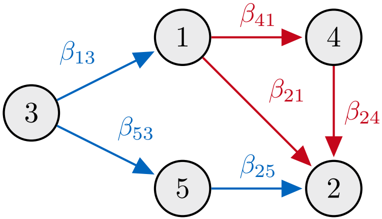

where $\varepsilon_j \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_j^2)$

Identifiabilty

Question: Given an observational distribution P_{Σ_0} from a set of possible distributions $\{P_\Sigma : \Sigma \in \mathcal{M}\}$, can we recover the (causal) target of interest $\mathcal{C}(i \to j)$?

Identifiabilty

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Question: Given an observational distribution P_{Σ_0} from a set of possible distributions $\{P_\Sigma : \Sigma \in \mathcal{M}\}$, can we recover the (causal) target of interest $\mathcal{C}(i \to j)$?

- Each DAG generates (sub-)set $\mathcal{M}(G) \subseteq \mathcal{M}$.
- What happens if $\Sigma_0 \in \mathcal{M}(G^1) \cap \mathcal{M}(G^2)$? $\varphi(G^1, \Sigma_0) = \varphi(G^2, \Sigma_0)$?

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- Each DAG generates (sub-)set $\mathcal{M}(G) \subseteq \mathcal{M}$.
- What happens if $\Sigma_0 \in \mathcal{M}(G^1) \cap \mathcal{M}(G^2)$? $\varphi(G^1, \Sigma_0) = \varphi(G^2, \Sigma_0)$?
- Only need to consider all complete DAGs $\mathcal{G}(d)$.

Identifiability
Arbitrary error variances

Linear Gaussian SCM with arbitrary error variances: $\mathcal{M} = \bigcup_{G \in \mathcal{G}(d)} \mathcal{M}(G)$, where

$$\mathcal{M}(G) = \left\{ \Sigma \in \text{PD}(d) : \exists B \in \mathbb{R}^G, \omega \in \mathbb{R}^d \text{ with } \Sigma = (I_d - B)^{-1} \text{diag}(\omega) (I_d - B)^{-T} \right\}$$

Theorem (see, e.g., Pearl (2009)). *The total causal effect $\mathcal{C}(i \to j)$ is not generically identifiable under the assumption of an underlying linear Gaussian SCM.*

Identifiability
Equal error variances

Linear Gaussian SCM with partially equal error variances: $\mathcal{M}_{EV} = \bigcup_{G \in \mathcal{G}(d)} \mathcal{M}_{EV}(G)$, where

$$\mathcal{M}_{EV}(G) = \Big\{\Sigma \in \text{PD}(d) : \exists \omega > 0 \text{ with } \omega = \Sigma_{k,k|p(k)} \quad \forall\, k = 1, \dots, d\Big\}.$$

Theorem (Peters and Bühlmann (2014))**.** *The total causal effect $\mathcal{C}(i \to j)$ is globally identifiable under the assumption of an underlying linear Gaussian SCM with equal error variances.*

Identifiability
Partially equal error variances

Linear Gaussian SCM with partially equal error variances: $\mathcal{M}_{PEV} = \bigcup_{G \in \mathcal{G}(d)} \mathcal{M}_{PEV}(G)$, where

$$\mathcal{M}_{PEV}(G) = \left\{ \Sigma \in \mathsf{PD}(d) : \Sigma_{i,i|p(i)} = \Sigma_{j,j|p(j)} \right\}.$$

Identifiability

Partially equal error variances

Linear Gaussian SCM with partially equal error variances: $\mathcal{M}_{PEV} = \bigcup_{G \in \mathcal{G}(d)} \mathcal{M}_{PEV}(G)$, where

$$\mathcal{M}_{PEV}(G) = \left\{ \Sigma \in \mathsf{PD}(d) : \Sigma_{i,i|p(i)} = \Sigma_{j,j|p(j)} \right\}.$$

Theorem. *The total causal effect $\mathcal{C}(i \to j)$ is generically identifiable under the assumption of an underlying linear Gaussian SCM with equal error variances among i and j.*

Causal Inference under Structure Uncertainty

Question: Given observational data from P_{Σ_0} out of a set of possible distributions $\{P_\Sigma : \Sigma \in \mathcal{M}\}$, can we construct a confidence region for $\mathcal{C}(i \to j)$?

Causal Inference under Structure Uncertainty

Question: Given observational data from P_{Σ_0} out of a set of possible distributions $\{P_\Sigma : \Sigma \in \mathcal{M}\}$, can we construct a confidence region for $\mathcal{C}(i \to j)$?

- Statistical uncertainty in the numerical size of the effect.
- Structural uncertainty in the underlying causal graph.
- Causal uncertainty due to equivalence of multiple models.

Causal Inference under Structure Uncertainty

Question: Given observational data from P_{Σ_0} out of a set of possible distributions $\{P_\Sigma : \Sigma \in \mathcal{M}\}$, can we construct a confidence region for $\mathcal{C}(i \to j)$?

- **Main Idea:** Use test inversion.
- Employ Dual-LRT for the arising testing problem, that is, for all $\psi \in \mathbb{R}$ and $G \in \mathcal{G}(d)$ invert joint test of structure and effect size:

$$\mathrm{H}_0^{(\psi)}(G) : \Sigma \in \mathcal{M}_\psi(G) \quad \text{against} \quad \mathrm{H}_1 : \Sigma \in \mathcal{M} \backslash \mathcal{M}_\psi(G).$$

Causal Inference under Structure Uncertainty
Arbitrary error variances

Theorem. *An asymptotic $(1-\alpha)$-confidence set for the total causal effect $\mathcal{C}(i \to j)$ under the assumption of an underlying linear Gaussian SCM is given by*

$$\bigcup_{G \in \mathcal{G}(d) \,:\, D(G) \geq 0} [L(G), U(G)] \bigcup \{0\},$$

where

$$L(G) := \frac{-(\widehat{\Sigma}^{-1})_{j,i|d(i)\setminus\{j\}} - \sqrt{D(G)}}{(\widehat{\Sigma}^{-1})_{j,j|d(i)\setminus\{j\}}}, \qquad U(G) := \frac{-(\widehat{\Sigma}^{-1})_{i,j|d(i)\setminus\{j\}} + \sqrt{D(G)}}{(\widehat{\Sigma}^{-1})_{j,j|d(i)\setminus\{j\}}}.$$

Causal Inference under Structure Uncertainty

Partially equal error variances

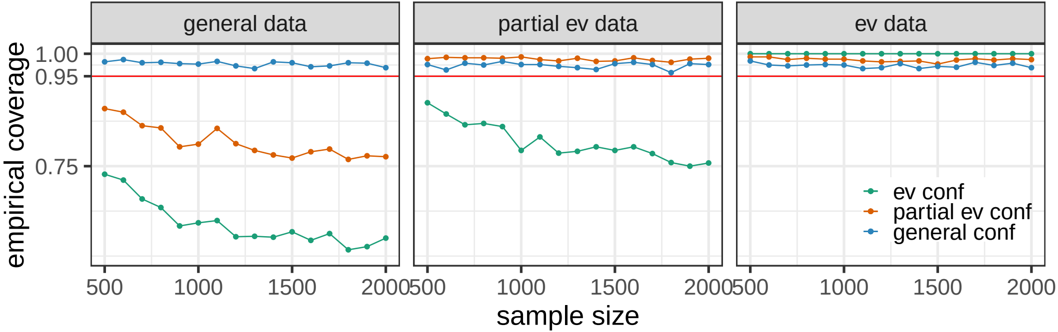
Theorem. *An asymptotic* $(1-\alpha)$*-confidence set for the total causal effect* $\mathcal{C}(i \to j)$ *under the assumption of an underlying linear Gaussian SCM with (partially) equal error variance among* i *and* j *is given by*

$$\bigcup_{G \in \mathcal{G}(d)\,:\,D(G) \geq 0} [L(G), U(G)] \bigcup \{0 : Z \leq K \exp\left(\frac{\chi^2_{1,1-\alpha}}{2n}\right)\},$$

where

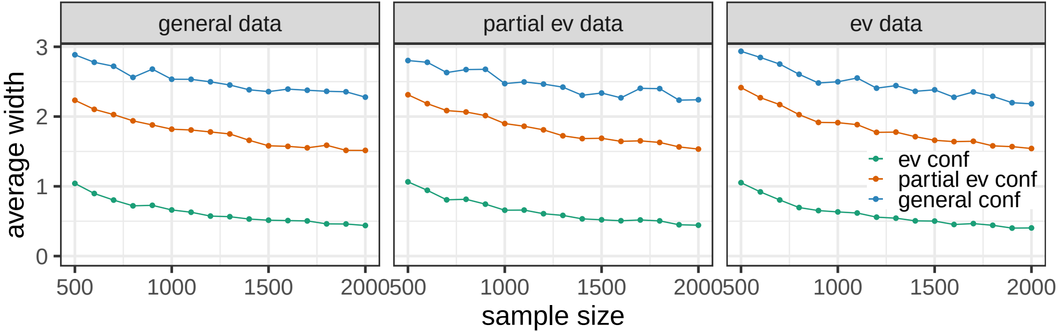
$$L(G) := \frac{-(\widehat{\Sigma}^{-1})_{j,i|d(i)\setminus\{j\}} - \sqrt{D(G)}}{(\widehat{\Sigma}^{-1})_{j,j|d(i)\setminus\{j\}}}, \qquad U(G) := \frac{-(\widehat{\Sigma}^{-1})_{i,j|d(i)\setminus\{j\}} + \sqrt{D(G)}}{(\widehat{\Sigma}^{-1})_{j,j|d(i)\setminus\{j\}}}.$$

Simulations



Empirical coverage of 95%-confidence intervals for the total causal effect.

Simulations



Mean width of 95%-confidence intervals for the total causal effect.

Conclusion

- The total causal effect is generically identifiable under partial homoscedasticity.
- Closed-form solution for constructing confidence regions for total causal effects that:
  - account for **causal structure uncertainty**
  - as well as **statistical uncertainty** about the numerical size of the effect.